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Complex string solutions of the self-dual Yang–Mills equations

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Abstract. We present, using the Atiyah–Ward construction, a new class of non-singular complex solutions of the self-dual Yang–Mills equations, dimensionally reduced to the plane.

1. Introduction

In recent years there has been much progress in our understanding of the self-dual Yang–Mills equations on compactified Euclidean 4-space S^4, corresponding to instantons (Belavin *et al* 1975), and in Euclidean 3-space $\mathbb{R}^3$ corresponding to monopoles in the Prasad–Sommerfield limit (Bogomolnyi 1976 and Prasad and Sommerfield 1975). Three different constructions (Ward 1977, Atiyah and Ward 1977, Atiyah *et al* 1978, Nahm 1981 and Forgács *et al* 1981) have been developed to deal with these solutions, each with its own advantages in different situations.

Despite these successes, some problems remain, in particular the following.

(a) Little is known about how the above three constructions are related; clearly, more examples would help to shed light on this problem.

(b) There is an obvious gap to be filled, and that is the construction and physical interpretation of non-singular self-dual gauge fields dimensionally reduced to Euclidean 2-space $\mathbb{R}^2$.

In this paper, we address ourselves to question (b). We shall consider the dimensional reduction of a pure SU(2) Yang–Mills theory from four to two dimensions. In analogy with the case of monopoles, the components A_3, A_4 of the gauge potentials now become effective Higgs fields Φ_1, Φ_2 say, and the Yang–Mills Lagrangian reduces to

$$\mathscr{L} = \tfrac{1}{2}\|B\|^2 + \tfrac{1}{2}\|D_i\Phi_1\|^2 + \tfrac{1}{2}\|D_i\Phi_2\|^2 + \tfrac{1}{2}\|[\Phi_1, \Phi_2]\|^2$$

where $i = 1, 2$, $B = F_{12} = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]$, and $\|B\|^2 = -2\,\mathrm{Tr}\,B^2$, etc. So we have an SU(2) gauge field interacting with two adjoint Higgs fields with an extra interaction term

$$V(\Phi_1, \Phi_2) = \tfrac{1}{2}\|[\Phi_1, \Phi_2]\|^2.$$

This model has also been considered by Nielsen and Olesen (1973), and Lohe (1977).

We shall in fact construct classical solutions with the non-trivial boundary conditions

$$\|\Phi_1\|^2 \sim a^2, \qquad \|\Phi_2\|^2 \sim b^2 \qquad \text{as } (x_1, x_2) \to \infty \tag{1.1}$$

so we should regard the potential V as the Bogomolny limit $\lambda \to 0$ of the potential

$$V'(\Phi_1, \Phi_2) = \tfrac{1}{2}\|[\Phi_1, \Phi_2]\|^2 + \tfrac{1}{4}\lambda(\|\Phi\|^2 - c^2)^2$$

where $\|\Phi\|^2 = \|\Phi_1\|^2 + \|\Phi_2\|^2$, $c^2 = a^2 + b^2$.

The self-duality equations reduce to the Bogomolny equations

$$B = [\Phi_1, \Phi_2], \qquad D_1\Phi_1 + D_2\Phi_2 = 0, \qquad D_1\Phi_2 - D_2\Phi_1 = 0. \tag{1.2}$$

We shall construct solutions of (1.2) which are essentially the two-dimensional version of monopoles—we call these solutions 'complex strings' or 'voidons', for reasons soon to be made clear. The physical interpretation of these solutions is still somewhat tentative; however, in order to attempt to reproduce at the semi-classical level the successes of string models of hadrons, and string-like behaviour at strong coupling in lattice gauge theories, it seems natural to us to seek solutions of the classical equations of motion which depend on only two of the space coordinates. There are some apparent problems with this point of view—first there is the question of what, if anything, makes the boundary conditions (1.1) appear naturally. Some authors have pointed out (Ambjørn *et al* 1979, Ambjørn and Olesen 1980a, b and 't Hooft 1981) that pure non-abelian gauge degrees of freedom may give rise to some analogue of the Higgs mechanism. From this point of view, our boundary conditions appear quite natural—presumably, in a semi-classical approximation, one would have to integrate over all the solutions for all possible values of the characteristic mass $c = (a^2 + b^2)^{1/2}$ and perhaps argue that the most significant contribution comes from those solutions with c of order Λ_{QCD}.

Secondly, up to now, solutions of the self-dual Yang–Mills equations have been sought which are both real and non-singular. However, it is easy to prove that self-dual gauge fields in two dimensions which are both real and non-singular, and which arise from the Bäcklund transformation approach to the Atiyah–Ward construction, must satisfy

$$\|\Phi\|^2 \equiv c^2, \qquad \text{energy density } \mathscr{E} \equiv 0$$

so that the Higgs vacuum fills the whole of space. So, in order to find non-singular self-dual fields in two dimensions from the Atiyah–Ward Bäcklund transformations, we have to drop the reality condition, i.e. we must seek solutions with gauge group $SL(2, \mathbb{C})$, the complexification of $SU(2)$. Real singular string-like solutions have been constructed by Saclioglu (1981).

The condition which forces non-singular solutions to be either strictly complex or the vacuum is that the total energy density per unit length, or equivalently, the total action in the x_1x_2 plane must vanish. Our solutions represent soliton-like enhancements of positive energy density in a sea of negative energy density in such a way that the total energy density integrates out to zero—hence the term 'voidon'.† The fact that the total action vanishes is an advantage in the semi-classical approximation. It is now fairly well established (Richard and Rouet 1981a, b, Lapedes and Mottala 1982, Abbott 1982, Abbott and Zakrzewski 1983) that in the saddle point approximation to functional integrals, both real and complex saddle points are of equal importance. Moreover, the contribution of any particular saddle point is damped by a factor $\exp(-S/g^2\hbar)$, where S is its total action. Therefore, it is not unreasonable to expect that complex saddle points of zero total action provide the leading contributions in a semi-classical approximation to the functional integral, giving a contribution of roughly the same order as the real vacuum. Note further that the usual argument that

† This term was first coined for complex solutions of zero total action by Dolan (1978).

non-perturbative effects are small of order $\exp(-1/g^2\hbar)$ clearly does not apply to saddle points of zero total action.

2. The Atiyah–Ward construction

In this section we review the most basic details of the Atiyah–Ward construction, and establish notation, and some preliminary results. Further details may be found in Corrigan *et al* (1978), Corrigan and Goddard (1981), Prasad (1981) and Prasad and Rossi (1980).

We use Yang variables

$$y=\tfrac{1}{2}\sqrt{2}(x_1+\mathrm{i}x_2), \qquad \bar{y}=\tfrac{1}{2}\sqrt{2}(x_1-\mathrm{i}x_2),$$
$$z=\tfrac{1}{2}\sqrt{2}(x_3-\mathrm{i}x_4), \qquad \bar{z}=\tfrac{1}{2}\sqrt{2}(x_3+\mathrm{i}x_4),$$

and we define twistor variables $(\omega, \pi)(\omega, \pi$ complex 2-spinors) by

$$\omega=x\pi, \qquad x=x_4+\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{\sigma}, \qquad \pi\neq 0.$$

We define coordinates (μ, ν, ζ) of projective twistor space $\mathbb{C}P^3$ by $\zeta=\pi_1/\pi_2$, $2\mu=\omega_1/\pi_1$, $2\nu=\omega_2/\pi_2$, i.e.

$$\begin{aligned}\mu&=\tfrac{1}{2}\mathrm{i}[(x_3-\mathrm{i}x_4)+(x_1-\mathrm{i}x_2)/\zeta],\\ \nu&=\tfrac{1}{2}\mathrm{i}[(x_1+\mathrm{i}x_2)\zeta-(x_3+\mathrm{i}x_4)].\end{aligned} \tag{2.1}$$

Gauge potentials in the nth Atiyah–Ward ansatz a_n may be described explicitly as follows. Define a Δ-chain to be a sequence of functions $\Delta_n(x)$, $n\in\mathbb{Z}$ satisfying the Cauchy–Riemann like conditions

$$\partial_y\Delta_k=-\partial_{\bar{z}}\Delta_{k+1}, \qquad \partial_z\Delta_k=\partial_{\bar{y}}\Delta_{k+1}. \tag{2.2}$$

Form the determinants

$$D^{(n)}=\begin{vmatrix}\Delta_0 & \Delta_1 & \Delta_2 & \cdots & \Delta_{n-1}\\ \Delta_{-1} & \Delta_0 & \Delta_1 & & \\ \Delta_{-2} & \Delta_{-1} & \Delta_0 & & \vdots\\ \vdots & & & \ddots & \\ \Delta_{-n+1} & \cdots & & & \Delta_0\end{vmatrix}.$$

Then a solution of Yang's R-gauge equations in the nth ansatz a_n is given by

$$\phi_n=D^{(n)}/D^{(n-1)},$$

$$\rho_n=\frac{(-1)^n}{D^{(n-1)}}\begin{vmatrix}\Delta_{-1} & \Delta_0 & \Delta_1 & & \Delta_{n-2}\\ \Delta_{-2} & \Delta_{-1} & \Delta_0 & & \vdots\\ \vdots & & & \ddots & \vdots\\ \Delta_{-n} & \cdots & & & \Delta_{-1}\end{vmatrix},$$

$$\bar{\rho}_n=\frac{(-1)^{n-1}}{D^{(n-1)}}\begin{vmatrix}\Delta_1 & \Delta_2 & \Delta_3 & & \Delta_n\\ \Delta_0 & \Delta_1 & \Delta_2 & & \vdots\\ \vdots & & & \ddots & \vdots\\ \Delta_{-n+2} & \cdots & & & \Delta_1\end{vmatrix},$$

and non-singularity is equivalent to the non-vanishing of the determinant $D^{(n)}$.

The Δ-chain equations are precisely the conditions that the generating function

$$\Delta(x, \zeta) = \sum_{n=-\infty}^{\infty} \Delta_n(x)\zeta^{-n} \tag{2.3}$$

depends only on the coordinates (μ, ν, ζ) of $\mathbb{C}P^3$. Conversely, the Laurent decomposition of any analytic function $\Delta(\mu, \nu, \zeta)$ gives a Δ-chain.

The patching matrix of the associated vector bundle over $\mathbb{C}P^3$ is given in the nth ansatz by

$$g(\mu, \nu, \zeta) = \begin{pmatrix} \zeta^n & \Delta(\mu, \nu, \zeta) \\ 0 & \zeta^{-n} \end{pmatrix}.$$

We shall construct our solutions by explicitly integrating the Δ-chain equations, and then we shall deduce the form of the patching functions. It is possible to work in the other direction, but this seems to us less well motivated.

In the construction of monopole solutions, the reduction to $\mathbb{R}^3$ is performed by demanding that the Δ-chain take the form

$$\Delta_n(x) = \exp(\mathrm{i}ax_4)\tilde{\Delta}_n(x_1, x_2, x_3). \tag{2.4}$$

By analogy, we shall reduce to $\mathbb{R}^2$ by demanding that the Δ-chain take the form

$$\Delta_n(x) = \exp[\mathrm{i}(ax_3 + bx_4)]\tilde{\Delta}_n(x_1, x_2). \tag{2.5}$$

Note that the Δ-chain equations imply that each Δ_n satisfies the four-dimensional Laplace equation, and hence that each $\tilde{\Delta}_n$ satisfies the two-dimensional Helmholtz equation

$$\nabla^2\tilde{\Delta}_n = c^2\tilde{\Delta}_n.$$

The following result, due to Prasad, guarantees that (2.4) gives the correct boundary conditions for monopole solutions of charge n in the nth ansatz.

Theorem. Suppose the Δ-chain satisfies (2.4). Then the Higgs field $\Phi = A_4$ and the energy density $\mathscr{E}$ in the nth ansatz are given by

$$\|\Phi\|^2 = a^2 - \nabla^2 \ln D^{(n)},$$
$$\mathscr{E} = -\tfrac{1}{2}\nabla^2\|\Phi\|^2 = -\tfrac{1}{2}\nabla^2\nabla^2 \ln D^{(n)}.$$

This result has two immediate corollaries for the ansatz (2.5).

Corollary 1. If the Δ-chain satisfies (2.5), then the Higgs field $\Phi_1 = A_3$, $\Phi_2 = A_4$, and the energy density $\mathscr{E}$ are given in the nth ansatz by

$$\|\Phi_1\|^2 = a^2 - \nabla^2 \ln D^{(n)}, \qquad \|\Phi_2\|^2 = b^2 - \nabla^2 \ln D^{(n)},$$
$$\therefore \|\Phi\|^2 = \|\Phi_1\|^2 + \|\Phi_2\|^2 = c^2 - 2\nabla^2 \ln D^{(n)},$$
$$\mathscr{E} = -\tfrac{1}{2}\nabla^2\|\Phi_1\|^2 = -\tfrac{1}{2}\nabla^2\|\Phi_2\|^2 = -\tfrac{1}{2}\nabla^2\nabla^2 \ln D^{(n)}. \tag{2.6}$$

Note that it is always possible to choose one of a, b to vanish, by performing an appropriate rotation in the x_3x_4 plane.

Corollary 2. Suppose the Δ-chain satisfies (2.5). Then, if (r, θ) are cylindrical polar coordinates in the x_1x_2 plane, we have in the nth ansatz

$$\|\Phi_1\|^2 \sim a^2 - nc/r, \qquad \|\Phi_2\|^2 \sim b^2 - nc/r,$$
$$\|\Phi\|^2 \sim c^2(1 - 2n/cr), \qquad \text{as } r \to \infty,$$
$$\mathscr{E} \sim -nc/2r^3, \qquad \text{as } r \to \infty. \tag{2.7}$$

If $E(R)$ is the total energy per unit length, or the total action, in a disc of radius R centred at the origin, then

$$E(R) \sim n\pi c/R \to 0 \qquad \text{as } R \to \infty. \tag{2.8}$$

Hence the total energy per unit length in the x_1x_2 plane is zero. If moreover the gauge field is real, then

$$\|\Phi\|^2 \equiv c^2 \qquad \text{and} \qquad \mathscr{E} \equiv 0.$$

Proof. We have $\Delta_k(x) = \exp[\mathrm{i}(ax_3 + bx_4)]\tilde{\Delta}_k(x_1, x_2)$ where $\nabla^2\tilde{\Delta}_k = c^2\Delta_k$. Hence

$$\tilde{\Delta}_k = \mathrm{e}^{cr} f_n(\theta)/\sqrt{cr},$$
$$D^{(n)} \sim \exp[\mathrm{i}(ax_3 + bx_4)](\mathrm{e}^{ncr}/(cr)^{n/2})\delta_n(\theta) \qquad \text{as } r \to \infty.$$

So provided the field is non-singular, i.e. $D^{(n)}$ does not vanish, we have

$$\ln D^{(n)} \sim ncr + \mathrm{O}(\log r),$$
$$\nabla^2 \ln D^{(n)} \sim nc/r, \qquad \nabla^2\nabla^2 \ln D^{(n)} \sim nc/r^2,$$

and (2.7) follows immediately. Also

$$E(R) = \int_{|x| \leqslant R} \mathrm{d}^2x\, \mathscr{E} = \tfrac{1}{2}\int_{|x| \leqslant R} \mathrm{d}^2x\, \nabla^2\|\Phi_1\|^2$$
$$= \tfrac{1}{2}\int_{|x| = R} \mathrm{d}\boldsymbol{n} \cdot \nabla\|\Phi_1\|^2 \sim n\pi c/R \qquad \text{as } R \to \infty.$$

For real gauge fields, we have $\mathscr{E} \geqslant 0$ on $\mathbb{R}^2$, since the Killing form of a compact group is positive definite. So

$$\int_{\mathbb{R}^2} \mathrm{d}^2x\, \mathscr{E}(x) = 0 \Rightarrow \mathscr{E} \equiv 0 \text{ on } \mathbb{R}^2$$

and

$$\mathscr{E} = -\tfrac{1}{4}\nabla^2\|\Phi\|^2 = 0, \qquad \|\Phi\|^2 \sim c^2 \text{ as } r \to \infty$$

implies

$$\|\Phi\|^2 \equiv c^2 \text{ on } \mathbb{R}^2.$$

It is tempting to conclude from the above result that all self-dual fields reduced to $\mathbb{R}^2$ have zero total action. However, the proof assumes that the field is constructed from the Bäcklund transformation approach to the Atiyah–Ward construction, which in turn assumes that the patching matrix is equivalent to one in upper triangular form. This latter condition is known to be true for instantons and monopoles, but still requires proof for fields satisfying our boundary conditions†.

† This fact was pointed out to me by R S Ward.

In the next section, we shall construct axially symmetric solutions in each Atiyah–Ward ansatz. We shall call a solution occurring in the nth ansatz a solution of 'charge' n, though there is no obvious definition of topological charge; the dependence on n seems to be entirely contained in the boundary conditions (2.7).

3. The axially symmetric N-string solutions

We construct the axially symmetric N-string solutions following Prasad's construction of the axially symmetric N-monopole solutions (Prasad 1981, Prasad and Rossi 1980). We find that, as is the case for monopoles, for $N > 1$ the energy density is concentrated in an annulus centred at the origin, with the radius of the annulus increasing as N increases. The construction of the axially symmetric N-strings is, in contrast, much simpler than the corresponding construction for monopoles; the axially symmetric N-string solution is obtained simply by applying the BI transformations N times to the a_1 ansatz for the single string. This class of solutions was also found, independently, by S Rouhani.

Recall the construction of the BPS monopole in the a_1 ansatz. One defines

$$\phi_1 = \exp(\mathrm{i}ax_4)\Lambda_0(x_1, x_2, x_3), \qquad \nabla^2\Lambda_0 = a^2\Lambda_0$$

and to obtain a spherically symmetric, non-singular field configuration, one chooses Λ_0 to be the spherically symmetric non-vanishing solution of the three-dimensional Helmholtz equation, namely

$$\Lambda_0 = \sinh ar/r.$$

Similarly, to construct the 1-string solution, we define

$$\phi_1 = \exp[\mathrm{i}(ax_3 + bx_4)]\Lambda_0(x_1, x_2), \qquad \nabla^2\Lambda_0 = c^2\Lambda_0$$

and choose Λ_0 to be the non-vanishing axially symmetric solution of the two-dimensional Helmholtz equation, namely

$$\Lambda_0 = I_0(cr)$$

where I_0 is the modified Bessel function of zeroth order.

This gives us the field configuration

$$A_1 = \frac{\mathrm{i}}{2}\begin{pmatrix} c\dfrac{I_1(cr)}{I_0(cr)}\dfrac{x_2}{r} & a+\mathrm{i}b \\ -a+\mathrm{i}b & -c\dfrac{I_1(cr)}{I_0(cr)}\dfrac{x_2}{r} \end{pmatrix},$$

$$A_2 = -\frac{\mathrm{i}}{2}\begin{pmatrix} c\dfrac{I_1(cr)}{I_0(cr)}\dfrac{x_1}{r} & -b+\mathrm{i}a \\ b+\mathrm{i}a & -c\dfrac{I_1(cr)}{I_0(cr)}\dfrac{x_1}{r} \end{pmatrix},$$

$$A_3 = \frac{1}{2}\begin{pmatrix} b & -c\dfrac{I_1(cr)}{I_0(cr)}\dfrac{x_1-\mathrm{i}x_2}{r} \\ c\dfrac{I_1(cr)}{I_0(cr)}\dfrac{x_1+\mathrm{i}x_2}{r} & -b \end{pmatrix},$$

$$A_4 = -\frac{1}{2}\begin{pmatrix} a & ic\dfrac{I_1(cr)}{I_0(cr)}\dfrac{x_1 - ix_2}{r} \\ ic\dfrac{I_1(cr)}{I_0(cr)}\dfrac{x_1 + ix_2}{r} & -a \end{pmatrix}.$$

Using (2.6), with $D^{(1)} = I_0(cr)$ we find

$$\|\Phi_1\|^2 = c^2[I_1(cr)^2/I_0(cr)^2] - b^2, \qquad \|\Phi_2\|^2 = c^2[I_1(cr)^2/I_0(cr)^2] - a^2,$$
$$\|\Phi\|^2 = c^2[2[I_2(cr)^2/I_0(cr)^2] - 1].$$

But $I_1(x)/I_0(x)$ increases monotonically from zero to one as x increases from zero to infinity; hence $\|\Phi\|^2$ increases monotonically from $-c^2$ to c^2 as r increases from zero to infinity. This is in contrast to the case of the BPS monopole, where the minimum of the norm of the Higgs field is zero, situated at the location of the monopole; indeed $\|\Phi\|^2$ is only allowed to become negative for strictly complex solutions, since the Killing form is positive definite on the Lie algebra of SU(2).

Integrating the Δ-chain equations, starting with the above choice for a_1 (see the appendix), we find, in the a_n ansatz:

$$D^{(n)} = \exp[in(ax_3 + bx_4)]\tilde{D}^{(n)},$$
$$\phi_n = \exp[i(ax_3 + bx_4)]\tilde{D}^{(n)}/\tilde{D}^{(n-1)},$$

$$\rho_n = \frac{(-1)^n}{\tilde{D}^{(n-1)}}\exp[i(ax_3 + bx_4)]\xi^{-n}\begin{vmatrix} I_1 & I_0 & I_1 & \dots I_{n-2} \\ I_2 & I_1 & I_0 & \\ I_3 & I_2 & I_1 & \vdots \\ \vdots & & \ddots & \\ I_n & & \dots & I_1 \end{vmatrix},$$

$$\bar{\rho}_n = \frac{(-1)^{n-1}}{\tilde{D}^{(n-1)}}\exp[i(ax_3 + bx_4)]\xi^{n}\begin{vmatrix} I_1 & I_2 & I_3 \dots I_n \\ I_0 & I_1 & I_2 & \\ I_1 & I_0 & I_1 & \vdots \\ \vdots & & \ddots & \\ I_{n-2} & & \dots & I_1 \end{vmatrix},$$

where $I_k = I_k(cr)$, $\gamma = a + ib$, $\xi = (i\gamma r/\sqrt{2}\,cy) \in U(1)$, and

$$\tilde{D}^{(n)} = \begin{vmatrix} I_0 & I_1 & I_2 \dots I_{n-1} \\ I_1 & I_0 & I_1 & \\ I_2 & I_1 & I_0 & \vdots \\ \vdots & & \ddots & \\ I_{n-1} & & \dots & I_0 \end{vmatrix}.$$

Non-singularity of the 1st and 2nd ansätze are automatic, since

$$\tilde{D}^{(1)}(r) = I(cr) > 0, \qquad \tilde{D}^{(2)}(r) = I_0(cr)^2 - I_1(cr)^2 > 0.$$

Non-singularity for higher n has been checked numerically for $n \leqslant 5$. It is found that $\ln \tilde{D}^{(n)}$ increases monotonically with r, and assumes its asymptotic linear form

$$\ln D^{(n)} \sim ncr$$

at around $cr = 2n$, i.e. at a radius of about $2n$ units of characteristic length.

In figure 1, $\|\Phi\|^2(r)$, $E(R)$ and $\mathscr{E}(r)$ are plotted numerically for the axially symmetric N-string ($N = 1$ to 5), with the characteristic length $c^{-1} = 1$. Note that $\|\Phi\|^2$ always takes the minimum $-c^2$ at $r = 0$. This is easily checked analytically; a power series

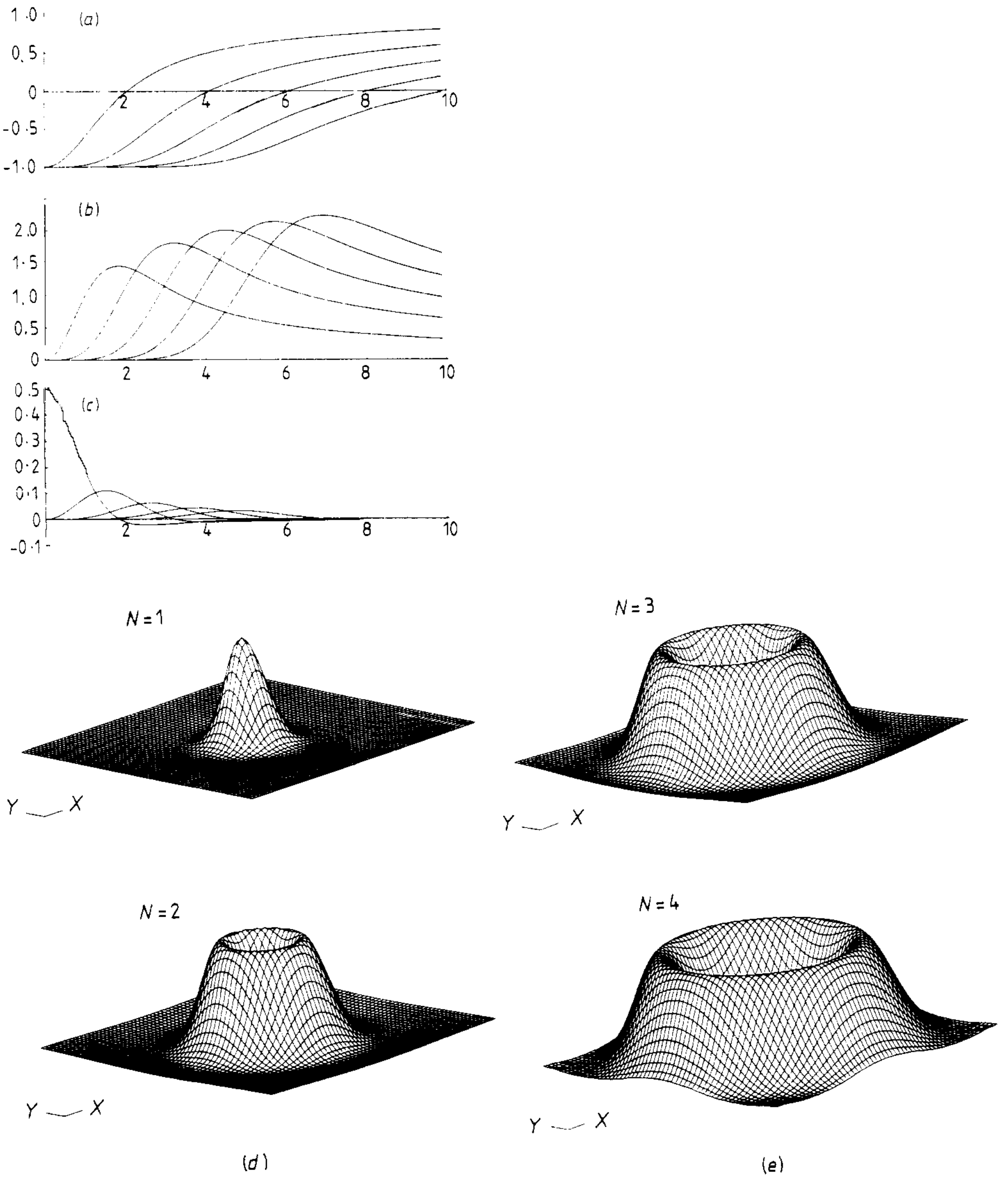

Figure 1. (a) $\|\Phi\|^2$ for axially symmetric N-string, $N = 1$–5. (b) $E(R)$ for axially symmetric N-string, $N = 1$–5. (c) Energy density $\mathscr{E}$ of axially symmetric N-string, $N = 1$–5. (d) Energy density of axially symmetric 1- and 2-strings. (e) Energy density of axially symmetric 3- and 4-strings.

expansion gives

$$\tilde{D}^{(n)} = 1 + \tfrac{1}{4}c^2r^2 + O(r^4) \Rightarrow \|\Phi\|^2 = -c^2 + O(r^2).$$

Note also that the region of positive energy density is concentrated in an annulus around the origin, which is always contained in the region where $\|\Phi\|^2 \leqslant 0$. A simple estimate of the size of this region is given by the asymptotic formula (2.11); we have

$$\|\Phi\|^2 \sim c^2(1 - 2n/cr), \qquad \therefore \|\Phi\|^2 = 0 \text{ at } r \approx 2n/c$$

in excellent agreement with the more precise numerical results.

So, the n-string differs appreciably from the vacuum in a disc of radius $2n$ units of characteristic length, centred at the origin; it is natural to identify this as the 'core' region of the n-string.

We defer the proof of non-singularity of the axially symmetric n-string to § 5, where it is proved in a more general context.

4. The separated 2-string solution

In the previous section, we found that the behaviour of the axially symmetric N-string solution was more or less analogous to the behaviour of the axially symmetric N-monopole solution. In contrast, we shall find in this section that the behaviour of the separated 2-string solution is remarkably different to that of the separated 2-monopole solution.

Again, the actual construction of the solutions is much simpler than in the monopole case. We take

$$\phi_1 = \exp[\mathrm{i}(ax_3 + bx_4)]\Lambda_0(x_1, x_2)$$

where Λ_0 is chosen to be the sum of two Λ_0's for axially symmetric 1-strings situated at different points, i.e.

$$\Lambda_0 = \alpha I_0(cr_1) + \beta I_0(cr_2)$$

where α, β are positive real constants, equivalent up to a scale factor, and

$$r_1 = [(x_1 - h_1)^2 + (x_2 - k_1)^2]^{1/2}, \qquad r_2 = [(x_1 - h_2)^2 + (x_2 - k_2)^2]^{1/2}.$$

Using equation (A3), this gives us, in the kth ansatz,

$$\begin{aligned}
\Delta_0 &= \exp \mathrm{i}(ax_3 + bx_4)(\alpha I_0(cr_1) + \beta I_0(cr_2)),\\
\Delta_k &= \exp \mathrm{i}(ax_3 + bx_4)(\alpha \xi_1^k I_k(cr_1) + \beta \xi_2^k I_k(cr_2)),\\
\Delta_{-k} &= \exp \mathrm{i}(ax_3 + bx_4)(\alpha \xi_1^{-k} I_k(cr_1) + \beta \xi_2^{-k} I_k(cr_2)),
\end{aligned} \tag{4.1}$$

where

$$\xi_1 = \mathrm{i}\gamma r_1/\sqrt{2}c(y - y_1), \qquad \xi_2 = \mathrm{i}\gamma r_2/\sqrt{2}c(y - y_2) \in U(1)$$

and

$$y_1 = \tfrac{1}{2}\sqrt{2}(h_1 + \mathrm{i}k_1), \qquad y_2 = \tfrac{1}{2}\sqrt{2}(h_2 + \mathrm{i}k_2).$$

We shall work in the a_2 ansatz. We have, expanding the determinant,

$$D^{(2)} = \begin{vmatrix} \Delta_0 & \Delta_1 \\ \Delta_{-1} & \Delta_0 \end{vmatrix} = \exp[2\mathrm{i}(ax_3 + bx_4)]\tilde{D}^{(2)},$$

where

$$\tilde{D}^{(2)} = \alpha^2[I_0(cr_1)^2 - I_1(cr_1)^2] + \beta^2[I_0(cr_2)^2 - I(cr_2)^2] + \alpha\beta[2I_0(cr_1)I_0(cr_2) - (\xi_2\xi_1^{-1} + \xi_1\xi_2^{-1})I_1(cr_1)I_1(cr_2)].$$

But, since ξ_1, $\xi_2 \in \mathrm{U}(1)$, we have

$$\xi_2\xi_1^{-1} + \xi_1\xi_2^{-1} = \xi_1\bar{\xi}_2 + \xi_2\bar{\xi}_1 = 2\,\mathrm{Re}\,\xi_1\bar{\xi}_2 = 2\,\mathrm{Re}\,\xi_1\xi_2^{-1}$$

and $\xi_1\xi_2^{-1} \in \mathrm{U}(1)$ implies $|\mathrm{Re}\,\xi_1\xi_2^{-1}| \leqslant 1$. Hence

$$\begin{aligned}\tilde{D}^{(2)} &= \alpha^2[I_0(cr_1)^2 - I_1(cr_1)^2] + \beta^2[I_0(cr_2)^2 - I_1(cr_2)^2] \\ &\quad + 2\alpha\beta[I_0(cr_1)I_0(cr_2) - \mathrm{Re}(\xi_1\xi_2^{-1})I_1(cr_1)I_1(cr_2)] \\ &\geqslant \alpha^2[I_0(cr_1)^2 - I_1(cr_1)^2] + \beta^2[I_0(cr_2)^2 - I_1(cr_2)^2] \\ &\quad + 2\alpha\beta[I_0(cr_1)I_0(cr_2) - I_1(cr_1)I_1(cr_2)] \\ &> 0,\end{aligned}$$

since $I_0(x) > I_1(x) \geqslant 0$ for all x.

Hence we have proved our solution is non-singular.

Without loss of generality, let us consider displacements along the x_1 axis, centred at the origin, i.e. let

$$r_1 = [(x_1 + h)^2 + x_2^2]^{1/2}, \qquad r_2 = [(x_1 - h)^2 + x_2^2]^{1/2}.$$

Then $\mathrm{Re}\,\xi_1\bar{\xi}_2 = (r^2 - h^2)/r_1r_2$, so

$$\tilde{D}^{(2)} = \alpha^2[I_0(cr_1)^2 - I_1(cr_1)^2] + \beta^2[I_0(cr_2)^2 - I_1(cr_2)^2] + \alpha\beta\{2I_0(cr_1)I_0(cr_2) - [(r^2 - h^2)/r_1r_2]I_1(cr_1)I_1(cr_2)\}. \tag{4.2}$$

We have studied the separated 2-string solution (4.2) numerically, for various values of the separation parameter h, with the characteristic length $c^{-1} = 1$. We define the physical locations of the strings to be the points at which $\|\Phi\|^2$ is a minimum. For values of h greater than about three units of characteristic length, the physical locations coincide with the peaks in the energy density; for values of h much less than three units of characteristic length, the ring-like structure is still evident, with the energy density being peaked at a distance greater than the physical separation.

$\|\Phi\|^2$ and $\mathscr{E}$ are plotted along the x_1 axis in figure 2 for $h = 2.5$, 5.0, 7.5, 10.0 and in figure 3 for $h = 20$, 30, 40, 50, with $\alpha = \beta = 1$. Note that $\|\Phi\|^2$ always takes the minimum value -1 at points $\pm h_{\text{phys}}$ on the x_1 axis; we call h_{phys} the physical separation parameter. We find numerically

$$h_{\text{phys}} \sim h \qquad \text{for } 0 \leqslant h \leqslant 0.6$$

and for larger values of h, we have

h	h_{phys}	$\mathscr{E}_{\max}$	$\mathscr{E}_{\min}$
1	0.9	0.16	−0.24
3	1.5	0.342	−0.167
10	2.0	0.687	−0.257
30	2.5	0.875	−0.296
50	2.7	0.925	−0.309
80	2.9	0.951	−0.316

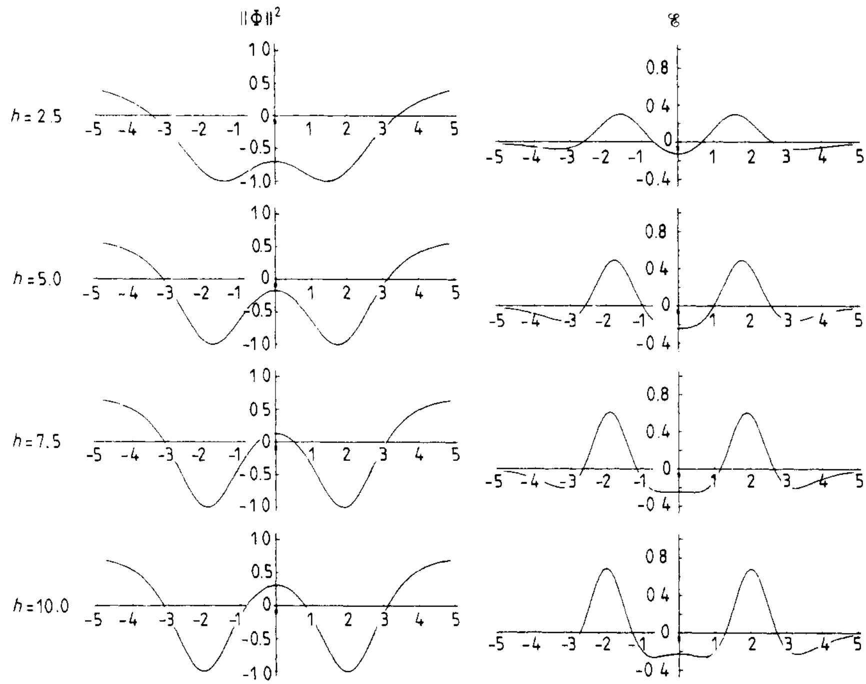

Figure 2. $\|\Phi\|^2$ and $\mathscr{E}$ for the separated 2-string solution with $h = 2.5$, 5.0, 7.5, 10.0.

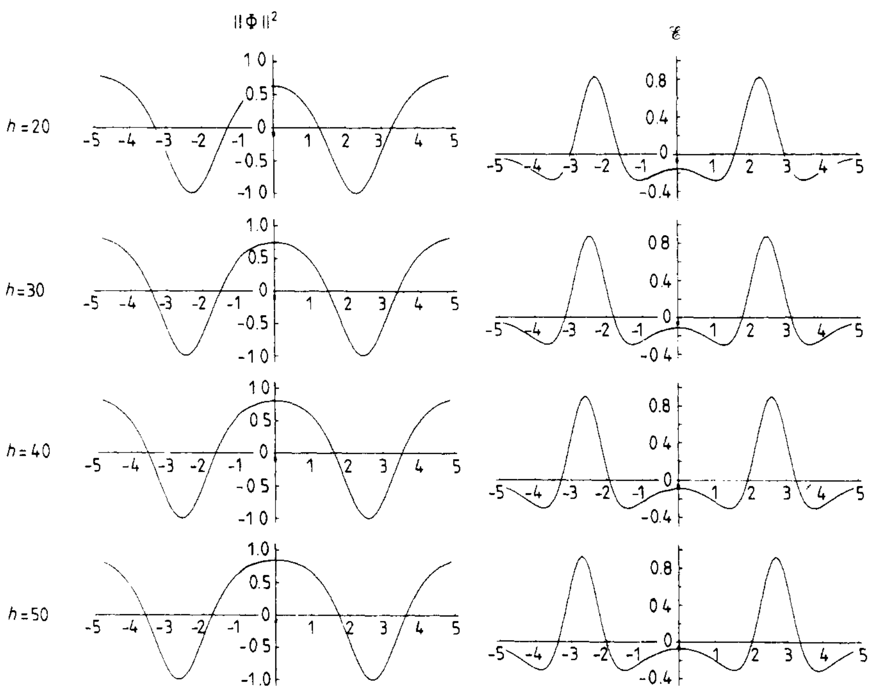

Figure 3. $\|\Phi\|^2$ and $\mathscr{E}$ for the separated 2-string solution with $h = 20$, 30, 40, 50.

So, for large h, h_{phys} varies extremely slowly with h. Also, note that the energy density profile of an isolated single string has

$$\mathscr{E}_{\max} = 0.5, \qquad \mathscr{E}_{\min} \simeq -0.02$$

so the energy density of the separated solution is clearly not converging to that of two isolated 1-strings; it is much more strongly peaked along the x_1 axis. This is in sharp contrast to the case of the separated 2-monopole solution, where the physical separation is always of the same order as the separation parameter (O'Raifeartaigh *et al* 1982), and the solution approaches the profiles of two isolated monopoles at large separation.

Variation of the parameter β/α does not spoil this behaviour; it has the effect of translating the profiles along the x_1 axis, with the actual separation of the strings remaining unchanged. Variation of β/α also has the curious effect of decreasing $\mathscr{E}_{\max}$ and raising the minimum of $\|\Phi\|^2$ above -1; this effect, however, is suppressed for large values of h.

The string profiles along the x_2 axis are even more surprising. Whereas the separation along the x_1 axis varies very slowly with h, there is an elongation along the x_2 axis, roughly of the same order as h. This behaviour is clearly shown in the contour graphs of figures 4 and 5, and the corresponding surface plots in figures 6 and 7. Thus we reach the remarkable conclusion that, at least within this class of separated solutions, it is impossible to approach the field configurations of two isolated 1-strings.

The above behaviour can be reproduced analytically. Set $c = 1$ and change notation to $x = x_1$, $y = x_2$. Recall the asymptotic expansions of $I_\nu(r)$:

$$I_\nu(r) \sim \frac{e^r}{(2\pi r)^{1/2}}\left(1 - \frac{4\nu^2 - 1}{8r}\right) \qquad \text{as } r \to \infty.$$

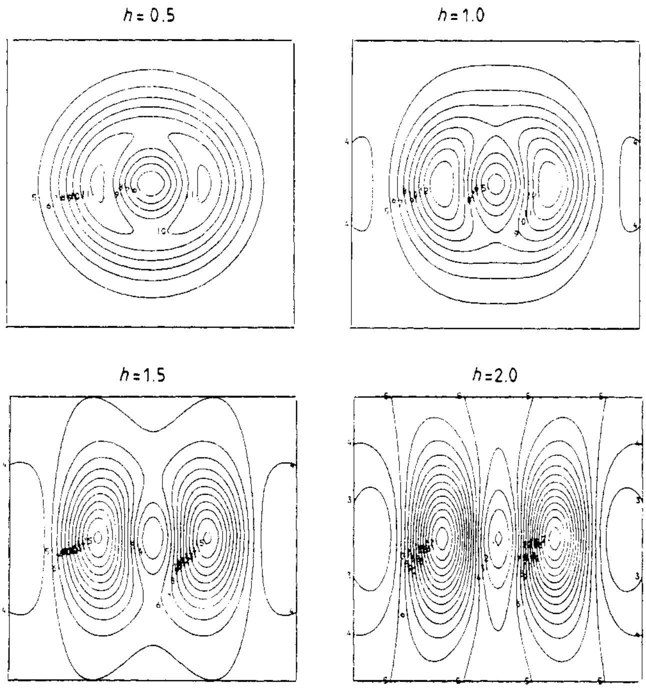

Figure 4. Contour plots of energy density for the separated 2-string at $h = 0.5, 1.0, 1.5, 2.0$. At contour n, $\mathscr{E} = -0.1 + 0.02n$.

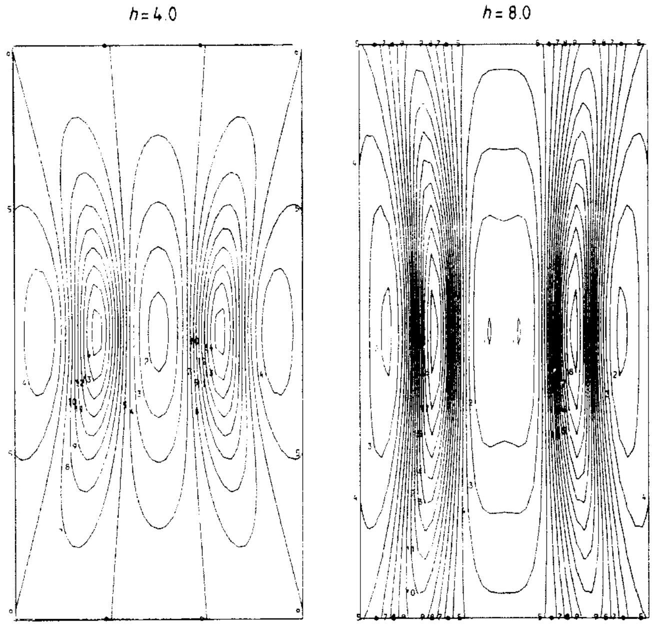

Figure 5. Contour plots of energy density for the separated 2-string, $h = 4.0$ and 8.0. At contour n, $\mathscr{E} = -0.3 + 0.05n$.

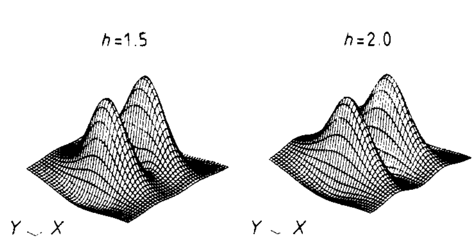

Figure 6. Surface plots of energy density for the separated 2-string with $h = (a)$ 0.5, (b) 1.0, (c) 1.5, (d) 2.0.

We use these to perform an asymptotic expansion for large h, or equivalently, for small x, y. More explicitly, assume $x, y = \mathrm{O}(\log h)$, and ignore terms of order x^2/h^2, y^2/h^2. After some algebra, we find

$$\tilde{D}^{(2)} \sim (\mathrm{e}^{2h}/2\pi h^2) \exp(y^2/h)[(\alpha\,\mathrm{e}^{x} - \beta\,\mathrm{e}^{-x})^2 + \alpha\beta(4h+1)], \qquad h \to \infty. \tag{4.3}$$

If we reparametrise α, β as follows

$$\alpha = \delta\,\mathrm{e}^{-x_0}, \beta = \delta\,\mathrm{e}^{x_0} \Rightarrow x_0 = \tfrac{1}{2} \ln \beta/\alpha$$

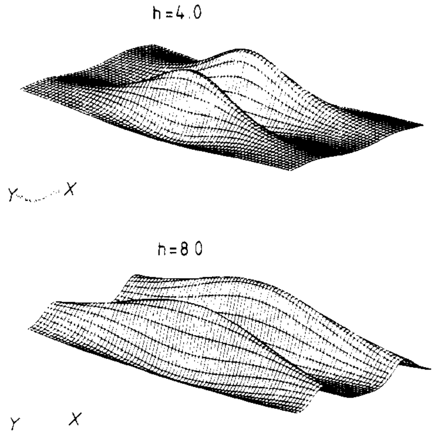

Figure 7. Surface plots of energy density for the separated 2-string with $h = 4.0$ and 8.0.

then (4.3) becomes

$$\tilde{D}^{(2)} \sim 2(\delta^2 e^{2h}/\pi h^2) \exp(y^2/h)[\sinh^2(x-x_0)+(h+\tfrac{1}{4})], \qquad h \to \infty \tag{4.4}$$

so, for large h, varying β/α merely translates the profiles a distance x_0 along the x axis. From (4.4) we obtain, setting $x_0 = 0$:

$$\ln \tilde{D}^{(2)} \sim y^2/h + \ln[\sinh^2 x + (h+\tfrac{1}{4})] + \text{constant},$$

$$\nabla^2 \ln \tilde{D}^{(2)} \sim 2h \frac{1+2\sinh^2 x}{(h+\sinh^2 x)^2}.$$

Hence we obtain

$$\|\Phi\|^2 \sim 1 - 4h \frac{1+2\sinh^2 x}{(h+\sinh^2 x)^2}, \qquad h \to \infty. \tag{4.5}$$

So, to this approximation, $\|\Phi\|^2$ is independent of y. So,

$$\partial\|\Phi\|^2/\partial y \sim 0 \text{ for } y = O(\log h), \qquad \text{as } h \to \infty.$$

Also, we easily deduce from (4.5) that

$$\|\Phi^2\|_{\min} \sim -1 \text{ at } \sinh^2 x \sim h, \qquad \text{as } h \to \infty.$$

Hence

$$\sinh^2 h_{\text{phys}} \sim h \qquad \text{as } h \to \infty,$$

i.e.

$$h_{\text{phys}} \sim \log 2\sqrt{h} \qquad \text{as } h \to \infty.$$

5. The separated N-string configurations and construction of patching functions

First let us consider the general form of the patching function for solutions with our dimensional reduction. We have

$$\Delta(x, \zeta) = \exp[\mathrm{i}(ax_3 + bx_4)] \sum_{n=-\infty}^{\infty} \tilde{\Delta}_n(x_1, x_2)\zeta^{-n}, \tag{*}$$

and in twistor variables (2.1) we have

$$\mu - \nu = \mathrm{i}x_3 + \tfrac{1}{2}\sqrt{2}\mathrm{i}(\bar{y}/\zeta - y\zeta),$$
$$\mu + \nu = x_4 + \tfrac{1}{2}\sqrt{2}\mathrm{i}(\bar{y}/\zeta + y\zeta).$$

We deduce that Δ is guaranteed to have the form (*) if, in twistor coordinates,

$$\Delta(\mu, \nu, \zeta) = \exp[a(\mu - \nu)] \exp[\mathrm{i}b(\mu + \nu)] f(\zeta) \tag{5.1}$$

where f is an arbitrary function analytic in an annular neighbourhood $1 - \varepsilon < |\zeta| < 1 + \varepsilon$. Of course, not all choices of f are guaranteed to give non-singular solutions.

Let us calculate the form of the patching function for the axially symmetric N-string configurations. Note that since these are all obtained by direct integration of the a_1 ansatz for the 1-string, they all have the same generating function Δ. Recall

$$\Delta_k = \exp[\mathrm{i}(ax_3 + bx_4)]\xi^k I_k(cr),$$
$$\xi = (\mathrm{i}\gamma/c)(x_1 - \mathrm{i}x_2)/r \in \mathrm{U}(1).$$

Hence we have

$$\Delta(x, \zeta) = \exp[\mathrm{i}(ax_3 + bx_4)] \sum_{k=-\infty}^{\infty} \left(\frac{\xi}{\zeta}\right)^k I_k(cr)$$

and using the generating function for Bessel functions

$$\exp[\tfrac{1}{2}z(t + 1/t)] = \sum_{k=-\infty}^{\infty} t^k I_k(z)$$

we immediately deduce

$$\Delta(x, \zeta) = \exp[\mathrm{i}(ax_3 + bx_4)] \exp[\tfrac{1}{2}cr(\xi/\zeta + \zeta/\xi)].$$

Inserting the expression for ξ, and rearranging the exponentials, we finally obtain

$$\Delta(x, \zeta) = \exp[a(\mu - \nu)] \exp[\mathrm{i}b(\mu + \nu)]. \tag{5.2}$$

So, comparing with (5.1), we see that the axially symmetric solutions have the simplest possible patching function, with $f(\zeta) \equiv 1$.

The obvious ansatz for separated N-string configurations which generalises that for the separated 2-string is to take

$$\phi_1 = \exp[\mathrm{i}(ax_3 + bx_4)]\Lambda_0(x_1, x_2),$$
$$\Lambda_0 = \sum_{l=1}^{n} \alpha_l I_0(cr_l), \qquad \alpha_l > 0,$$
$$r_l = [(x - x_1^{(l)})^2 + (x_2 - x_2^{(l)})^2]^{1/2},$$

and then integrate this up to the nth ansatz.

We find

$$\left.\begin{aligned} \Delta_k &= \exp[\mathrm{i}(ax_3 + bx_4)] \sum_{l=1}^{n} \alpha_l \xi_l^k I_k(cr_l), \\ \Delta_{-k} &= \exp[\mathrm{i}(ax_3 + bx_4)] \sum_{l=1}^{n} \alpha_l \xi_l^{-k} I_k(cr_l), \end{aligned}\right. \tag{5.3}$$

where

$$\xi_l = \mathrm{i}\gamma r_l/\sqrt{2}c(y - y_l) \in \mathrm{U}(1), \qquad y_l = \tfrac{1}{2}\sqrt{2}(x_1^{(l)} + \mathrm{i}x_2^{(l)}).$$

Let μ_l, ν_l be μ, ν evaluated at $y=y_l$, $z=0$, i.e.

$$\mu_l=\tfrac{1}{2}\sqrt{2}\mathrm{i}\bar{y}_l/\zeta, \qquad \nu_l=\tfrac{1}{2}\sqrt{2}\mathrm{i}y_l\zeta.$$

Then writing $\hat{\mu}_l=\mu-\mu_l$, $\hat{\nu}_l=\nu-\nu_l$, the generating function for (5.3) is given by

$$\begin{aligned}\Delta(x,\zeta)&=\sum_{l=1}^{n}\alpha_l\exp[a(\hat{\mu}_l-\hat{\nu}_l)+\mathrm{i}b(\hat{\mu}_l+\hat{\nu}_l)]\\&=\exp[a(\mu-\nu)]\exp[\mathrm{i}b(\mu+\nu)]f(\zeta)\end{aligned}$$

where

$$\begin{aligned}f(\zeta)&=\sum_{l=1}^{n}\alpha_l\exp[(\mu_l-\nu_l)+\mathrm{i}b(\mu_l+\nu_l)]\\&=\sum_{l=1}^{n}\alpha_l\exp\left[\frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\gamma}y_l\zeta-\frac{\gamma\bar{y}_l}{\zeta}\right)\right].\end{aligned}\tag{5.4}$$

Finally, let us prove that (5.3) gives a non-singular field configuration in the a_n ansatz. This will prove as a special case the non-singularity of the axially symmetric n-string configurations.

Let $D^{(n)}=\exp[\mathrm{i}n(ax_3+bx_4)]\tilde{D}^{(n)}$ be the determinant formed from (5.3). By analogy with the proof of positivity of $\tilde{D}^{(2)}$, we may rearrange the expression for $\tilde{D}^{(n)}$ so that it contains sums of complex conjugate pairs

$$\xi+\bar{\xi}=2\operatorname{Re}\xi, \qquad \xi\in\mathrm{U}(1).$$

These are bounded by ±2, and we easily deduce that it is sufficient to prove

$$\delta^{(n)}=\begin{vmatrix}I_0(r_1) & I_1(r_1) & I_2(r_1) & \dots & I_{n-1}(r_1)\\ I_1(r_2) & I_0(r_2) & I_1(r_2) & & \\ I_2(r_3) & I_1(r_3) & I_0(r_3) & & \vdots\\ \vdots & & & \ddots & \\ I_{n-1}(r_n) & & \dots & & I_0(r_n)\end{vmatrix}>0$$

for all values of $r_1,\dots,r_n$.

Proof. (Due to S Rouhani). Using the integral representation

$$I_n(r)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\exp(r\cos\theta+\mathrm{i}n\theta)$$

we obtain

$$\delta^{(n)}=\frac{1}{(2\pi)^n}\int_0^{2\pi}\mathrm{d}\theta_1\dots\mathrm{d}\theta_n\exp(r_1\cos\theta_1+\dots+r_n\cos\theta_n)\Theta^{(n)}$$

where

$$\Theta^{(n)}=\begin{vmatrix}1 & \cdots & \mathrm{e}^{\mathrm{i}\theta_1} & \mathrm{e}^{2\mathrm{i}\theta_1}\dots & \mathrm{e}^{(n-1)\mathrm{i}\theta_1}\\ \mathrm{e}^{-\mathrm{i}\theta_2} & & 1 & \mathrm{e}^{\mathrm{i}\theta_2} & \\ \mathrm{e}^{-2\mathrm{i}\theta_3} & & \mathrm{e}^{-\mathrm{i}\theta_3} & 1 & \vdots\\ \vdots & & & \ddots & \\ \mathrm{e}^{-(n-1)\theta_n} & & \dots & & 1\end{vmatrix}.$$

Since $\delta^{(n)}$ is real, the integration only picks up the real part of $\Theta^{(n)}$ so we have, using Weyl's character formula (Prasad 1981, Prasad and Rossi 1980),

$$\delta^{(n)}=\frac{1}{(2\pi)^n}\int_0^{2\pi} \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \exp(r_1 \cos\theta_1+\ldots+r_n\cos\theta_n)\prod_{i\neq j} 2\sin^2\left(\frac{\theta_i-\theta_j}{2}\right)>0,$$

since the integrand is positive.

It is most likely that we have not yet described all non-singular solutions. Note first that the separated n-string ansatz (5.3) is manifestly non-singular in the a_1 ansatz, where it would give rise to a 'distorted' 1-string solution. Moreover, the above proof of non-singularity actually applies in any ansatz a_k, not only $k=n$. Also, it is quite likely that there exist separated string solutions more akin to the case of monopoles, where the asymptotic field at large separation is like that of isolated 1-strings. Work is in progress to check these possibilities.

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Appendix. Atiyah–Ward ansätze for axially symmetric N-strings

In order to complete the description of the a_1 ansatz, and to integrate up to the a_n anzätze, we need to define a sequence of functions $\Lambda_k(r)$ by

$$\Lambda_0 \equiv I_0(cr),$$
$$\Lambda_k \equiv \bar{y}^{-1}\partial_y\Lambda_{k+1} \equiv y^{-1}\partial_{\bar{y}}\Lambda_{k+1} \equiv r^{-1}\partial_r\Lambda_{k+1},$$

i.e. $\Lambda_{k+1}=\int r\,\mathrm{d}r\Lambda_k$ (ignoring constants of integration).

Using a fundamental property of Bessel functions

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^{\nu}I_\nu(r))=r^{\nu-1}I_{\nu-1}(r), \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^{-\nu}I_\nu(r))=r^{-(\nu+1)}I_{\nu+1}(r) \qquad (*)$$

$$\Rightarrow \int (r\,\mathrm{d}r)r^{\nu}I_\nu(r)=r^{\nu+1}I_{\nu+1}(r),$$

we obtain

$$\Lambda_k(r)=(r^k/c^k)I_k(cr).$$

Define $\gamma=a+\mathrm{i}b$, $\bar{\gamma}=a-\mathrm{i}b$.

Lemma 1. The a_1 ansatz is solved by the functions

$$\phi_1=\exp[\mathrm{i}(ax_3+bx_4)]\Lambda_0,$$
$$\rho_1=\exp[\mathrm{i}(ax_3+bx_4)](\mathrm{i}\bar{\gamma}/\sqrt{2}\bar{y})\Lambda_1, \qquad \text{(A1)}$$
$$\bar{\rho}_1=\exp[\mathrm{i}(ax_3+bx_4)](\mathrm{i}\gamma/\sqrt{2}y)\Lambda_1.$$

Proof. We have $\rho_{1,y} = \phi_{1,\bar{z}}$, $\bar{\rho}_{1,\bar{y}} = \phi_{1,z}$, so

$$\rho_{1,y} = \{\exp[(\mathrm{i}/\sqrt{2})(\gamma z + \bar{\gamma}\bar{z})]\Lambda_0\}_{,z} = \exp[\mathrm{i}(ax_3 + bx_4)](\mathrm{i}\bar{\gamma}/\sqrt{2})\Lambda,$$

$$\therefore \rho_{1,y} = (\exp[\mathrm{i}(ax_3 + bx_4)](\mathrm{i}\bar{\gamma}/\sqrt{2}\bar{y})\Lambda_1)_{,y},$$

$$\bar{\rho}_{1,\bar{y}} = \{\exp[\mathrm{i}(ax_3 + bx_4)](\mathrm{i}\gamma/\sqrt{2}y)\Lambda_1\}_{,\bar{y}},$$

and these are clearly satisfied by (A1).

We must also check the other two equations $\rho_{1,z} = -\phi_{1,\bar{y}}$, $\bar{\rho}_{1,\bar{z}} = -\phi_{1,y}$. These are easily seen to be equivalent to

$$2\bar{y}\partial_{\bar{y}}\Lambda_0 \equiv 2y\partial_y\Lambda_0 \equiv r\partial\Lambda_0/\partial r = c^2\Lambda_1$$

which is an easy consequence of property (*) of Bessel functions.

Lemma 2. The a_k ansatz is solved by the functions

$$\begin{aligned} \Delta_0 &= \exp[\mathrm{i}(ax_3 + bx_4)\Lambda_0, \\ \Delta_{-k} &= \exp[\mathrm{i}(ax_3 + bx_4)](-\mathrm{i}\bar{\gamma}/\sqrt{2}\bar{y})^k\Lambda_k, \\ \Delta_k &= \exp[\mathrm{i}(ax_3 + bx_4)](\mathrm{i}\gamma/\sqrt{2}y)^k\Lambda_k. \end{aligned} \tag{A2}$$

Proof. Use induction on k; $k=1$ follows from lemma 1, since $\Delta_0 = \phi_1$, $\Delta_1 = \bar{\rho}_1$, $\Delta_{-1} = -\rho_1$.

First check the formula for Δ_k, for $k > 1$. Since $\partial_{\bar{y}}\Delta_{k+1} = \partial_z \Delta_k$, we have, by the inductive hypothesis,

$$\begin{aligned} \partial_{\bar{y}}\Delta_{k+1} &= \partial_z\{\exp[(\mathrm{i}/\sqrt{2})(\gamma z + \bar{\gamma}\bar{z})](\mathrm{i}\gamma/\sqrt{2}y)^k\Lambda_k\} \\ &= \{\exp[\mathrm{i}(ax_3 + bx_4)](\mathrm{i}\gamma/\sqrt{2}y)^{k+1}\Lambda_{k+1}\}_{,\bar{y}} \end{aligned}$$

and this is clearly satisfied by

$$\Delta_{k+1} = \exp[\mathrm{i}(ax_3 + bx_4)](\mathrm{i}\gamma/\sqrt{2}y)^{k+1}\Lambda_{k+1}.$$

We must also check the equation $\partial_y\Delta_k = -\partial_{\bar{z}}\Delta_{k+1}$. But

$$\partial_{\bar{z}}\Delta_{k+1} = \exp[\mathrm{i}(ax_3 + bx_4)](\mathrm{i}\gamma/\sqrt{2}y)^k(-c^2/2y)\Lambda_{k+1},$$

$$\partial_y\Delta_k = \exp[\mathrm{i}(ax_3 + bx_4)](\mathrm{i}\gamma/\sqrt{2}y)^k(1/2y)(2y\partial_y\Lambda_k + 2k\Lambda_k),$$

so it is sufficient to check that

$$r\partial_r\Lambda_k = 2k\Lambda_k + c^2\Lambda_{k+1}$$

which, again, is an easy consequence of property (*) of Bessel functions.

The formula for Δ_{-k} follows similarly.

Substituting $\Lambda_k = (r^k/c^k)I_k(cr)$ in (A2), we obtain

$$\begin{aligned} \Delta_k &= \exp[\mathrm{i}(ax_3 + bx_4)]\xi^k I_k(cr), \\ \Delta_{-k} &= \exp[\mathrm{i}(ax_3 + bx_4)]\xi^{-k} I_k(cr), \end{aligned} \tag{A3}$$

where

$$\xi = (\mathrm{i}\gamma r/\sqrt{2}cy) \in \mathrm{U}(1).$$

Using routine manipulations of determinants, (A3) gives us equations (3.1).

References

Abbott R B 1982 *University of Durham preprint* DTP-82/11
Abbott R B and Zakrzewski W J 1983 *University of Durham preprint* DTP-83/1
Ambjørn J, Nielsen N K and Olesen P 1979 *Nucl. Phys.* B **152** 75
Ambjørn J and Olesen P 1980a *Nucl Phys.* B **170** 60
—— 1980b *Nucl. Phys.* B **170** (FSI) 265
Atiyah M F, Drinfeld V G, Hitchin N J and Manin Yu I 1978 *Phys. Lett.* **65**A 185
Atiyah M F and Ward R S 1977 *Commun. Math. Phys.* **55** 117
Belavin A A, Polyakov A M, Schwartz A S and Tyupkin Yu S 1975 *Phys. Lett.* **59**B 85
Bogomolnyi E B 1976 *Sov. J. Nucl. Phys.* **24** 449
Corrigan E F, Fairlie D B, Yates R G and Goddard P 1978 *Commun. Math. Phys.* **58** 223
Corrigan E F and Goddard P 1981 *Commun. Math. Phys.* **80** 575
Dolan L 1978 *Phys. Lett.* **76**B 105
Forgács P, Horváth Z and Palla L 1981 *Phys. Lett.* **109**B 200
Lapedes A and Mottala E 1982 *Nucl. Phys.* B **203** 58
Lohe M A 1977 *Phys. Lett.* **70**B 325
Nahm W 1981 *Cern preprint* TH 3172-CE (*Talk given at the Int. Summer Inst. on Theor. Phys., Freiberg,* September 1981)
Nielsen H B and Olesen P 1973 *Nucl. Phys.* B **61** 45
O'Raifeartaigh L, Rouhani S and Singh L P 1982 *Nucl. Phys.* B **206** 137
Prasad M K 1981 *Commun. Math. Phys.* **80** 137
Prasad M K and Rossi P 1980 *MIT preprint* CTP 903
Prasad M K and Sommerfield C M 1975 *Phys. Rev. Lett.* **35** 760
Richard J L and Rouet A 1981a *Nucl. Phys.* B **183** 251
—— 1981b *Nucl. Phys.* B **185** 47
Saclioglu C 1981 *Nucl. Phys.* B **178** 361; *Bonn Preprint, BONN-HE-81-28*
't Hooft G 1981 *Nucl. Phys.* B **190** (FS3) 455
Ward R S 1977 *Phys. Lett.* **61**A 81